INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 7, 1315–1324 (1987)

A NEW APPROACH TO THE SOLUTION OF THE NAVIER–STOKES EQUATIONS

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SUMMARY

In the present paper a numerical algorithm is given for solving a standard problem in fluid dynamics, that of inviscid, irrotational, incompressible flow over an arbitrary symmetric profile. The purpose of the paper is to propose an alternative approach to solve certain fluid dynamic flows. This paper may be thought of as the first of a possible series of papers solving new and fundamental problems. In a sense, this new approach asks the question: what is the simplest and most efficient method of solving the problem considered by finite difference methods. It is believed that the following algorithm answers this question. Standard second-order finite difference techniques, such as SLOR and ADI, are used to solve numerically a mixed boundary value problem comprised of a pair of elliptic partial differential equations with constant coefficients.

KEY WORDS Numerical Solution Potential Flow

INTRODUCTION

A new field of research proposes an alternative method of solution to a class of fluid dynamic problems. The new approach uses as independent variables the streamlines $\psi =$ constant, of the flow under consideration and an arbitrary family of curves $\phi =$ constant. This approach is made possible by transforming the governing fluid flow equations from plane to curvilinear co-ordinates (ϕ, ψ), as accomplished by Martin.[1] The new system of partial differential equations is made determined; boundary conditions are then applied and the equations are solved on a square rectangular grid (Figure 1), composed of streamlines and, in the orthogonal case, potential curves. The boundary conditions are determined based on *a priori* knowledge of the location and direction of flow on specific streamlines, i.e. stagnation streamlines. The strength of the approach is the use of metrics: g_{11}, g_{12}, g_{22}, in the notation of differential geometry, as dependent variables. This provides for usually reliable results, provided that the grid size is chosen to be a 'good' size compared with the size of the profile being considered. The characteristic of goodness is, however, a difficult criterion to judge. If nothing else, this paper shows the weakness of second-order finite difference methods with regard to accuracy, and points out the need to use higher-order methods for such problems. However, accuracy is not the overall goal of this paper, but rather it is to present a different approach and show possibilities.

0271–2091/87/121315–10$05.00

Received December 1985
Revised April 1987

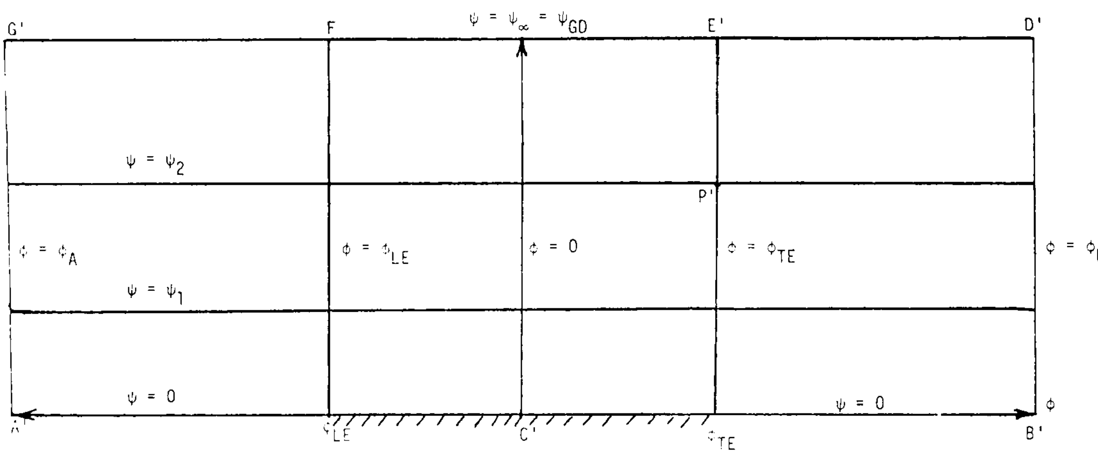

Figure 1. Computational grid

The objective of the present paper is the same as that of the streamline analysis or flux analysis method developed by Uchida,[2] based on the stream curvature method of Flügel.[3] Uchida's method begins with an orthogonal curvilinear co-ordinate system (α, β), independent of the flow variables, then approximates $\beta = \text{constant}$ near the curves of solution by a recursive relation from the continuity equation

$$\frac{\partial}{\partial \beta}\left(\frac{h_\alpha}{h_\beta}\frac{1}{\rho_m}\frac{\partial \psi_m}{\partial \beta}\right) = -\frac{\partial}{\partial \alpha}\left(\frac{h_\beta}{h_\alpha}\frac{1}{\rho_{m-1}}\frac{\partial \psi_{m-1}}{\partial \alpha}\right),$$

in which h_α and h_β correspond to the metrices $\sqrt{g_{11}}$ and $\sqrt{g_{22}}$, respectively. The above equation is an ordinary differential equation. Uchida used this method to solve for compressible, irrotational flow in a circular channel. Such a problem can be attempted by Martin's approach by reformulating the Navier–Stokes equations with variable density.

To begin with, consider the steady, plane, flow of a viscous, incompressible fluid over an arbitrary profile, $y = f(x)$ inserted at the origin. It is essential to construct a computational grid upon which the system of flow equations is discretized and solved, i.e. put into finite difference form. To avoid non-uniform grid spacing near the boundaries it is practical to construct numerically or algebraically a body conforming grid; this facilitates application of boundary conditions. The computational plane becomes a rectangular grid with uniform grid spacing in all directions. To generate a grid, one can solve elliptic partial differential equations (PDEs) such as[4]

$$(x_\varepsilon^2 + y_\varepsilon^2)x_{\eta\eta} - 2(x_\varepsilon x_\eta + y_\varepsilon y_\eta)x_{\varepsilon\eta} + (x_\eta^2 + y_\eta^2)x_{\varepsilon\varepsilon} = P(\varepsilon, \eta), \tag{1}$$

and similarly for y. $P(\varepsilon, \eta)$ is a control function prescribed or determined in the course of calculations to control grid spacing in various regions. A significant feature of (1) is its independence from the fluid flow equations. In the present method, many of the fundamental concepts of numerical grid generation are preserved and its techniques used extensively, i.e. solution of an elliptic PDE and construction of a computational grid with uniform grid spacing in both co-ordinate directions. However, as stated, there are several differences in this new approach: one being that the solution obtained is in terms of the metrices g_{11}, g_{22}, g_{12} of the first fundamental form of differential geometry

$$g_{11} = x_\phi^2 + y_\phi^2, \quad g_{12} = x_\phi x_\psi + y_\phi y_\psi, \quad g_{22} = x_\psi^2 + y_\psi^2, \tag{2}$$

where $g^2 = (g_{11}g_{22} - g_{12}^2)$ is the Jacobian of the transformation. A second difference is the use of Gauss's equation for vanishing plane curvature:

$$\frac{1}{g}\left[\left(\frac{g}{g_{11}}\Gamma_{11}^2\right)_\psi - \left(\frac{g}{g_{11}}\Gamma_{12}^2\right)_\phi\right] = 0, \tag{3}$$

in which Γ_{ij}^k denote Christoffel symbols, and

$$\Gamma_{11}^2 = \frac{-g_{12}(g_{11})_\phi + 2g_{11}(g_{12})_\phi - g_{11}(g_{11})_\psi}{2g^2}, \tag{4a}$$

$$\Gamma_{12}^2 = \frac{g_{11}(g_{22})_\phi - g_{12}(g_{11})_\psi}{2g^2}. \tag{4b}$$

Gauss's equation ensures uniqueness of the solution to within an arbitrary constant, since g_{ij} are coefficients of the first fundamental form,

$$dx^2 + dy^2 = g_{11}\,d\phi^2 + 2g_{12}\,d\phi\,d\psi + g_{22}\,d\psi^2, \tag{5}$$

if and only if they satisfy (3) for a curvilinear co-ordinate pair ϕ, ψ. Finally, a third difference is the absence of the forcing term in the present approach. This may constitute a disadvantage since other means, such as change of variable, must be employed to pack grid lines in regions of high gradient or where singularities exist, such as stagnation points. In the following sections, a finite difference technique is outlined that is general enough to solve for potential flows over arbitrary symmetric profiles without circulation. Flows with circulation, including those over an arbitrarily shaped profile at any angle of attack, can be attempted without a great deal of difficulty. This latter research area comprises a work in progress.

MARTIN'S FLOW EQUATIONS

The system of equations governing the steady, plane flow of a viscous, incompressible fluid is given by

$$(\rho u)_x + (\rho v)_y = 0, \tag{6a}$$

$$\rho(uu_x + vu_y) + p_x = \mu(u_{xx} + u_{yy}), \tag{6b}$$

$$\rho(uv_x + vv_y) + p_y = \mu(v_{xx} + v_{yy}), \tag{6c}$$

in which μ is the constant coefficient of viscosity and ρ the constant density. u, v and p are the components of velocity in the x and y directions and the pressure, respectively, all functions of x and y. By inspection (6) constitutes a system of three equations in three unknowns. By introducing the energy equation and vorticity equation,

$$h = \frac{\rho}{2}(u^2 + v^2) + p, \quad \omega = v_x - u_y,$$

respectively, the system of equations (6) can be written,[1] as a non-dimensional system of four equations in four unknowns u, v, h and ω:

$$u_x + v_y = 0, \tag{7a}$$

$$h_x - \omega v = -(Re)^{-1}\omega_y, \tag{7b}$$

$$h_y + \omega u = (Re)^{-1}\omega_x, \tag{7c}$$

$$\omega = v_x - u_y, \tag{7d}$$

where $Re = \rho U_{\infty} L/\mu$ is the Reynolds number. Introducing a stream function $\psi(x, y)$, we have $\psi_x = -v$, $\psi_y = u$. Furthermore, let $\phi(x, y)$ be a co-ordinate such that ψ, ϕ form a curvilinear net. Denote by α the angle of inclination of the curve $\psi(x, y) = \text{constant}$ to the x-axis. Then we have the following standard results of differential geometry:[5]

$$x_\phi = \sqrt{g_{11}} \cos\alpha, \quad y_\phi = \sqrt{g_{11}} \sin\alpha, \tag{8a}$$

$$x_\psi = \sqrt{g_{22}} \cos(\alpha + \theta), \quad y_\psi = \sqrt{g_{22}} \sin(\alpha + \theta). \tag{8b}$$

In (8) $\theta = \theta(\phi, \psi)$ is a function of position, the measure of the angle of intersection of the co-ordinate curves $\psi = \text{constant}$, $\phi = \text{constant}$. The integrability conditions, $x_{\psi\phi} = x_{\phi\psi}$ and $y_{\psi\phi} = y_{\phi\psi}$, yield

$$\alpha_\phi = \frac{g}{g_{11}} \Gamma^2_{11}, \quad \alpha_\psi = \frac{g}{g_{11}} \Gamma^2_{12}. \tag{9}$$

Likewise, integrability conditions for α yield (3), Gauss's equation. By introducing hodograph co-ordinates (q, δ), such that

$$u = q\cos\delta, \quad v = q\sin\delta$$

and using the equation of continuity, Martin[1] has shown the equivalence of the continuity equation to

$$q = \frac{\sqrt{g_{11}}}{g}. \tag{10a}$$

Subsequently, the flow equations (7) have been transformed by Martin to the following system of five equations, including continuity, in seven unknowns $g_{11}, g_{12}, g_{22}, q, h, p$ and ω:

$$g_{22} h_\phi - g_{12}(h_\psi + \omega) = -g(Re)^{-1}\omega_\psi, \tag{10b}$$

$$-g_{12} h_\phi - g_{11}(h_\psi + \omega) = g(Re)^{-1}\omega_\phi \quad \text{(Navier–Stokes)}, \tag{10c}$$

$$\omega = \frac{1}{g}\left[\left(\frac{g_{12}}{g}\right)_\phi - \left(\frac{g_{11}}{g}\right)_\psi\right] \quad \text{(vorticity)}. \tag{10d}$$

The energy equation can be written by (10a) as

$$h = \frac{g_{11}}{2g^2} + p. \tag{10e}$$

The addition of Gauss's equations and dropping of (10e) leaves a system of five equations in six unknowns. This requires some specification of the curves $\phi = \text{constant}$ so as to make the system determined. Then a numerical solution may be attempted.

BOUNDARY CONDITIONS

To remove a degree of arbitrariness in equations (10) one can choose a representation, such as

$$dx^2 + dy^2 = g_{11}(\phi, \psi)(d\phi^2 + 2\mathscr{K}\cos\theta\, d\phi\, d\psi + \mathscr{K}^2\, d\psi^2), \tag{11a}$$

$$\mathscr{K}^2 g_{11} = g_{22}, \quad g_{12} = \mathscr{K} g_{11}\cos\theta, \quad g = \mathscr{K} g_{11} \sin\theta. \tag{11b}$$

Usually, θ is chosen *a priori*, depending on the boundary conditions and hand. For constant $\mathscr{K}, \theta = \pi/2$, (11) is a conformal representation and the flow is necessarily irrotational by inspection of (10d). This is the type of flow we are concerned with in the present paper.

Conversely, if the flow is irrotational with constant θ, then inspection of (10d) with (11) yields

$$g_{11} = g_{22} z(\phi), z(\phi) > 0, \tag{12}$$

where z is an arbitrary function of ϕ. Since the flow is assumed uniform in upstream and downstream regions, it follows from (10a) that $z = \text{constant}$.

Taking $z(\phi) = \mathscr{K}^{-2} = \text{constant}$ in (12), and $\theta = \text{constant}$, and using this in (3) yields

$$\nabla_0^2[\ln(g_{11})] - 2\cos\theta\,[\ln(g_{11})]_{\psi\phi} = 0, \tag{13}$$

where

$$\nabla_0^2 = \mathscr{K}\frac{\partial^2}{\partial\phi^2} + \mathscr{K}^{-1}\frac{\partial^2}{\partial\psi^2}.$$

Likewise, by using (8) one can easily show that

$$\nabla_0^2[x] - 2\cos\theta\,[x]_{\psi\phi} = 0, \quad \nabla_0^2[y] - 2\cos\theta\,[y]_{\psi\phi} = 0. \tag{14}$$

For simplicity the upstream and downstream flows are considered to be described by linear functions of ψ and ϕ with constant A, B, C and D:

$$x_\infty(\phi,\psi) = A\phi + B\psi, y_\infty(\phi,\psi) = C\phi + D\psi. \tag{15}$$

From (11) and (15) we find

$$\mathscr{K}^{-2} = \left(\frac{A^2 + C^2}{B^2 + D^2}\right), \cot\theta = \left(\frac{AB + CD}{AD - CB}\right). \tag{16}$$

Thus, to describe the problem in the upstream and downstream regions one only has to specify the constants in (16) with non-vanishing denominators. Figure 2 illustrates the possible flow patterns for a symmetrical profile. It is clear that g_{11} and g_{22} are unknown on the profile; thus one approach

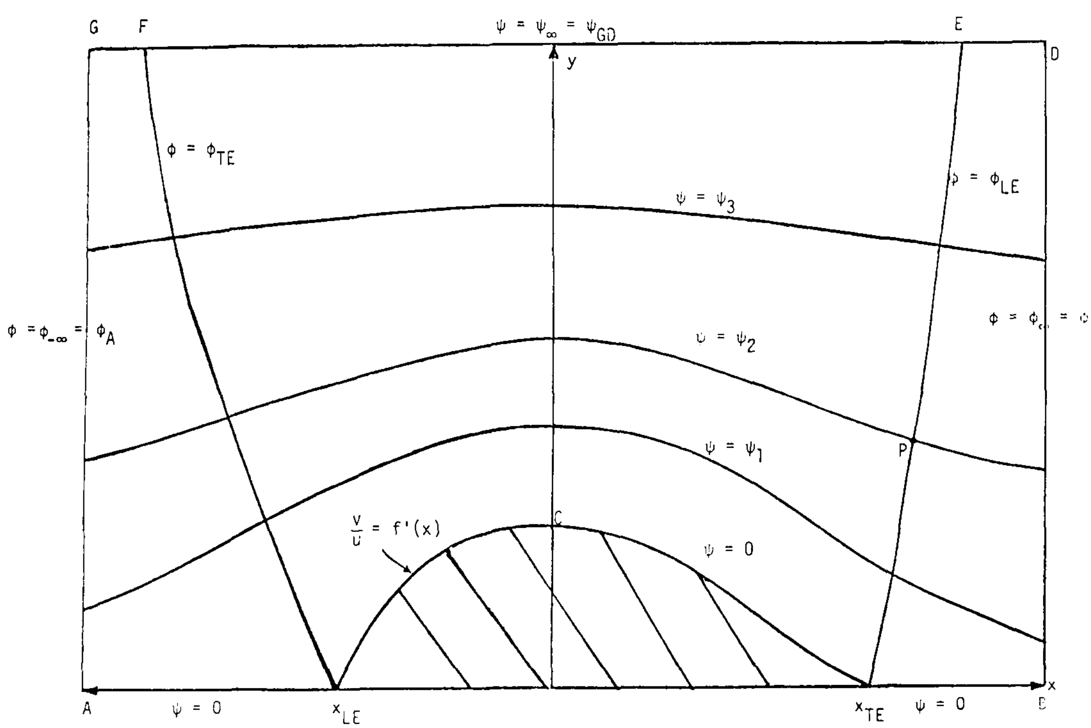

Figure 2. Physical plane

is to incorporate von Neumann boundary conditions on the profile and on the vanishing streamline in the symmetric case. For unknowns x and y these boundary conditions would come from equations (8), whereas boundary conditions for $\ln(g_{11}) = 2T$, say, are derived from (9) to yield, with use of (11b)

$$\alpha_\phi = T_\phi \cot\theta - \mathscr{K}^{-1} T_\psi \operatorname{cosec}\theta, \tag{17a}$$

$$\alpha_\psi = \mathscr{K} T_\phi \operatorname{cosec}\theta - T_\psi \cot\theta. \tag{17b}$$

The angle of inclination, α, of the velocity vector which is tangential to the profile for such flows is present in (17). α also satisfies an elliptic equation of the form (14); however, its values are only required on the profile surface. Given an aerofoil profile $y = f(x)$ it follows that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\alpha(x, y(x)) = f'(x). \tag{18}$$

Moreover, the flow tangency condition on the aerofoil surface is equivalent to

$$\frac{\mathrm{d}\psi}{\mathrm{d}x} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\tan\alpha(x, y(x)) = 0. \tag{19a}$$

By exploiting inverse relations between the co-ordinate pairs (x, y) and (ϕ, ψ) given by

$$\phi_x = y_\psi/g, \psi_x = -y_\phi/g, \phi_y = -x_\psi/g, \psi_y = x_\phi/g, \tag{19b}$$

one can show that the relations (9) are equivalent to

$$\tan\alpha = y_\phi/x_\phi = \frac{\mathrm{d}y}{\mathrm{d}x}. \tag{19c}$$

Thus, use of (18) is consistent with (9). By inspection of (18) it follows that values of $x(\phi, \psi)$ are required on the profile. The flow tangency condition is satisfied, so that on the profile ψ = constant may be assumed. Generally, an orthogonal grid is desirable for numerical computations, and the simplest possible boundary conditions are

$$x_\infty(\phi, \psi) = \phi, \quad y_\infty(\phi, \psi) = \psi, \tag{20}$$

for both upstream and downstream conditions. This yields $\mathscr{K} = 1$ and $\cot\theta = 0$ in (16). However, a suitable scale factor may be used in (20) with respect to y_∞ in order to accelerate convergence, i.e. $y_\infty(\phi, \psi) = D\psi$, D = constant. Use of (20) is suitable for the symmetric case. For the non-symmetric case it would desirable to have a non-zero incoming flow angle. From (19c)

$$\alpha_\infty(\phi, \psi) = \tan_\infty^{-1}\left(\frac{C}{A}\right), A \neq 0.$$

NUMERICAL ALGORITHM

We consider only the symmetric problem. After some numerical experimentation it was found that the following algorithm yielded satisfactory results with a minimum of numerical experimentation:

1. Construct a computational domain (Figure 1) with grid points represented by ordered integer pairs (i, j) such that

$$j_{\min} \leqslant j \leqslant j_{\max}, i_{\min} \leqslant i \leqslant i_{\max}.$$

   The computational domain is defined for our purposes as

$$(\phi, \psi) \subseteq [\phi_{\min}, \phi_{\max}] \times [0, \psi_{\max}].$$

The grid size is taken to be $j_{\max} - j_{\min} \times i_{\max} - i_{\min}$. The step sizes in the ψ and ϕ directions are given by

$$D\psi = \frac{\psi_{\max} - \psi_{\min}}{j_{\max} - j_{\min}}, \quad D\phi = \frac{\phi_{\max} - \phi_{\min}}{i_{\max} - i_{\min}}.$$

2. Choose constants A, B, C, D in (15), (16). These boundary conditions also determine T_∞, q_∞ since from (10a)

$$q_\infty = \left(\frac{\sqrt{g_{11}}}{g}\right)_\infty = (A^2 + C^2)^{0.5}/(AD - BC).$$

and since $T = (\ln g_{11})/2$, $T_\infty = [\ln(A^2 + C^2)]/2$.

3. Apply central differencing to (13) and to (14) for the equation involving x to obtain

$$\begin{aligned}(T_{i,j+1} - 2T_{i,j} + T_{i,j-1})/D\psi^2 + \mathscr{K}^2(T_{i+1,j} - 2T_{i,j} + T_{i-1,j})/D\phi^2 \\ = 2\mathscr{K}\cos\theta(T_{i+1,j+1} - T_{i-1,j+1} - T_{i-1,j+1} + T_{i-1,j-1})/(2D\phi D\psi).\end{aligned} \tag{21}$$

A similar equation exists for x. Boundary conditions in the ψ direction can be incorporated into (21) by the following equation ($i_{\min} = 1$):

$$(T_{\psi\psi})_{i,1} = \frac{2}{D\psi^2}[T_{i,2} - T_{i,1} - D\psi(T_\psi)_{i,1}] + o(D\psi^2) \tag{22a}$$

where, from (17a),

$$(T_\psi)_{i,1} = \mathscr{K}[\cos\theta(T_{i+1,j} - T_{i-1,j})/(2D\phi) - \sin\theta(\alpha_{i+1,j} - \alpha_{i-1,j})/(2D\phi)]. \tag{22b}$$

For x we have a similar pair of equations:

$$(x_{\psi\psi})_{i,1} = \frac{2}{D\psi^2}[x_{i,2} - x_{i,1} - D\psi(x_\psi)_{i,1}] + o(D\psi^2), \tag{23a}$$

where, from (8b),

$$(x_\psi)_{i,1} = \mathscr{K}\exp(T_{i,1})\cos(\theta + \alpha_{i,1}), \tag{23b}$$

in which $\mathscr{K}$ and θ are determined from (16) and α from

$$\alpha_{i,1} = \begin{matrix} \tan^{-1}[f'(x_{i,1})], & |x_{i,1}| < x_{LE} = x_{TE}, \\ 0, & \text{otherwise.} \end{matrix}$$

4. Use SLOR or ADI to solve iteratively for x and T as given in (22) and (23). This is done by using straightforward Gaussian elimination for tridiagonal matrices of the form

$$\mathrm{Trid}(-1, 2 + \rho, -1)\, T_i = \mathrm{rhs}_i, \quad i = i_{\min} + 1, \ldots, i_{\max} - 1;$$

similarily for j except that j begins at $j_{\min}$. In the above expression $\rho = 2$ in the case of SLOR and $\rho > 0$ for ADI. An acceleration parameter β is used in SLOR such that, at the kth iteration level,

$$T_{i,j}^{k+1} \leftarrow T_{i,j}^k + \beta(\mathrm{OLD}\ T_{i,j}^k - T_{i,j}^k),$$

where $\beta = 1$ for ADI and $\beta \geqslant 1$ for SLOR.

RESULTS

The test case was the semicircle of radius 1/2. This size of circle was chosen because of computational efficiency. The stream function and velocity potential in non-dimensional variables are given by [6]

$$\psi^* = \left(r^* - \frac{1}{r^*}\right)\sin\theta^*, \quad \phi^* = \left(r^* + \frac{1}{r^*}\right)\cos\theta^*, \tag{24}$$

where

$$\phi^* = \phi/(\rho\mathscr{V}_\infty a), \psi^* = \psi/(\rho\mathscr{V}_\infty a), \theta^* = \theta, r^* = r/a. \tag{25}$$

In (25) a is the circle radius, ρ the density and $\mathscr{V}_\infty$ the upstream and downstream speed. The test case profile was given by

$$y(x) = (0{\cdot}25 - x^2), \; x_{\mathrm{LE}} \leqslant x \leqslant x_{\mathrm{TE}}, \tag{26}$$

also, from (24)–(25)

$$\psi = \rho\mathscr{V}_\infty\left(r - \frac{a^2}{r}\right)\sin\theta, \phi = \rho\mathscr{V}_\infty\left(r + \frac{a^2}{r}\right)\cos\theta. \tag{27}$$

The radial and angular components of velocity are given by (taking $\mathscr{V}_\infty = \rho = 1$)

$$\psi_r = -v(r,\theta) = \left(1 + \frac{a^2}{r^2}\right)\sin\theta, \tag{28a}$$

$$\frac{\psi_\theta}{r} = u(r,\theta) = \left(1 - \frac{a^2}{r^2}\right)\cos\theta. \tag{28b}$$

From (28) the speed on the circle is given by

$$q(a,\theta) = 2\sin\theta. \tag{29}$$

By inspection of (27), (29) using $\alpha(\phi,0) = \theta - \pi/2$, we have

$$q(\phi,0) = 2\cos(\alpha(\phi,0)), \phi = -2a\sin(\alpha(\phi,0)). \tag{30}$$

Taking an orthogonal grid ($\theta = \pi/2$) in (10a), and using (12) yields

$$q(\phi,0) = 1/[\mathscr{K}\sqrt{g_{11}(\phi,0)}]. \tag{31}$$

Solving (30) and (31) for g_{11} gives

$$g_{11}(\phi,\theta) = \frac{1}{4\mathscr{K}^2(1 - \phi^2/4a^2)}, |\phi| < 2a, \tag{32}$$

on the aerofoil surface, with ϕ given by (30). An expression for $g_{11}(\phi,0)$ can also be determined for $|\phi| > 2a$, i.e. on the stagnation streamline but away from the surface. This can be determined from inverse relations between x, y and r, θ given by

$$r_\phi = \frac{\psi_\theta}{j}, \theta_\phi = -\frac{\psi_r}{j}, r_\psi = \frac{-\phi_\theta}{j}, \theta_\psi = \frac{\phi_r}{j}. \tag{33}$$

From (27) and (33), using the chain rule for derivatives

$$g_{11}(\phi,\psi) = r_\phi^2 + r^2\theta_\phi^2 = \frac{r^4}{r^4 - 2r^2a^2\cos 2\theta + a^4}. \tag{34}$$

It can also be shown that $g_{11} = g_{22}$ and $\mathscr{K} = 1$. Using $\theta = 0, \pi$ in (34) we have

$$g_{11}(\phi, 0) = \frac{r^4}{(r^2 - a^2)^2}, \; r > a. \tag{35}$$

To determine r as function of ϕ set $\theta = 0$, π in (27) to obtain a quadratic in r:

$$r^2 \pm \phi r + a^2 = 0. \tag{36}$$

Solving (36) yields

$$r = \pm 0{\cdot}5[\phi + \sqrt{(\phi^2 - 4a^2)}], \tag{37}$$

in which the positive sign is taken if $\phi > 2a$ and the negative sign taken if $\phi < -2a$. Equations (35) and (37) yield g_{11} off the surface of the aerofoil:

$$g_{11}(\phi, 0) = \frac{1}{[1 - (a/r)^2]^2}, \; r > a, \tag{38}$$

in which r is given by (37). The metric g_{11} given in (32), (38) can be used as a standard with which to compare the numerical solution. On the stagnation streamline $x_{i,1}$ is determined from the numerical solution rather than (37) in the comparison because an accurate relation between x and ϕ is difficult to determine. A grid size of 101×66 was taken with $x_{max} = -x_{min} = 2$, $y_{max} = 4{\cdot}5033$ and $y_{min} = 0$. CPU time was 10–40 minutes on a Cyber 835 depending on the value of the ratio of the grid size $D\psi/D\phi$. Convergence is fastest, it is found, when the ratio of grid sizes is roughly $\sqrt{3}$, and using ADI. Figures 3 and 4 show metric and speed profiles plotted at values of x. Agreement is first-order accurate over the profile and more exaggerated when the metric is inverted. To achieve a more accurate solution would require not necessarily a larger grid but a higher order method. In

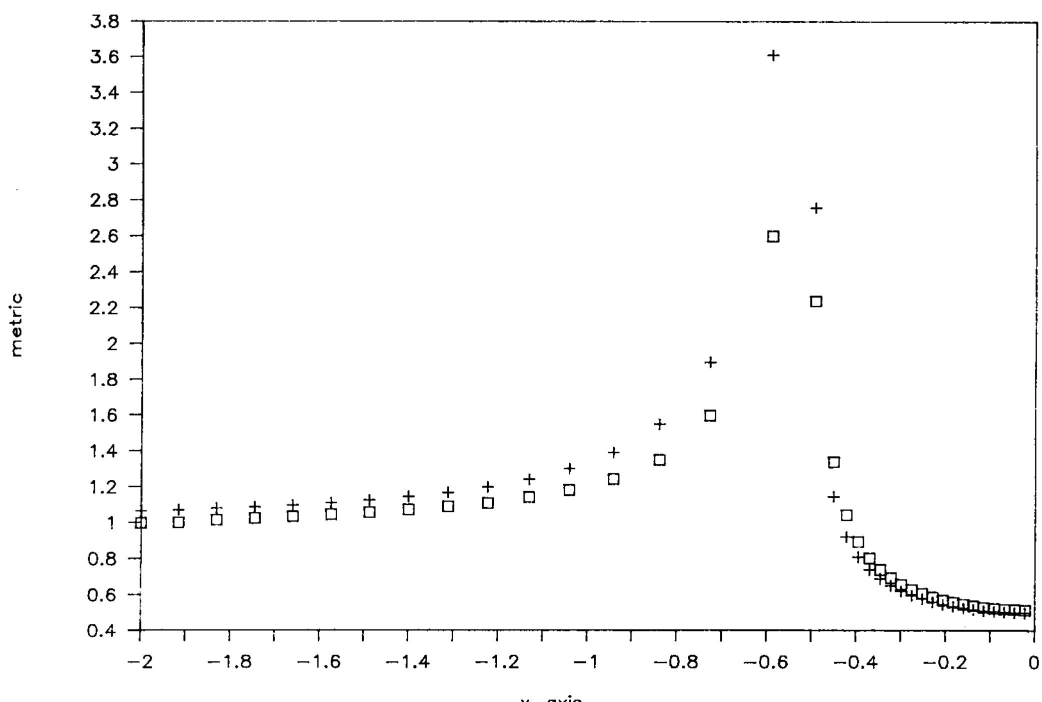

Figure 3. Metric VS. X: □ numerical solution; + exact solution

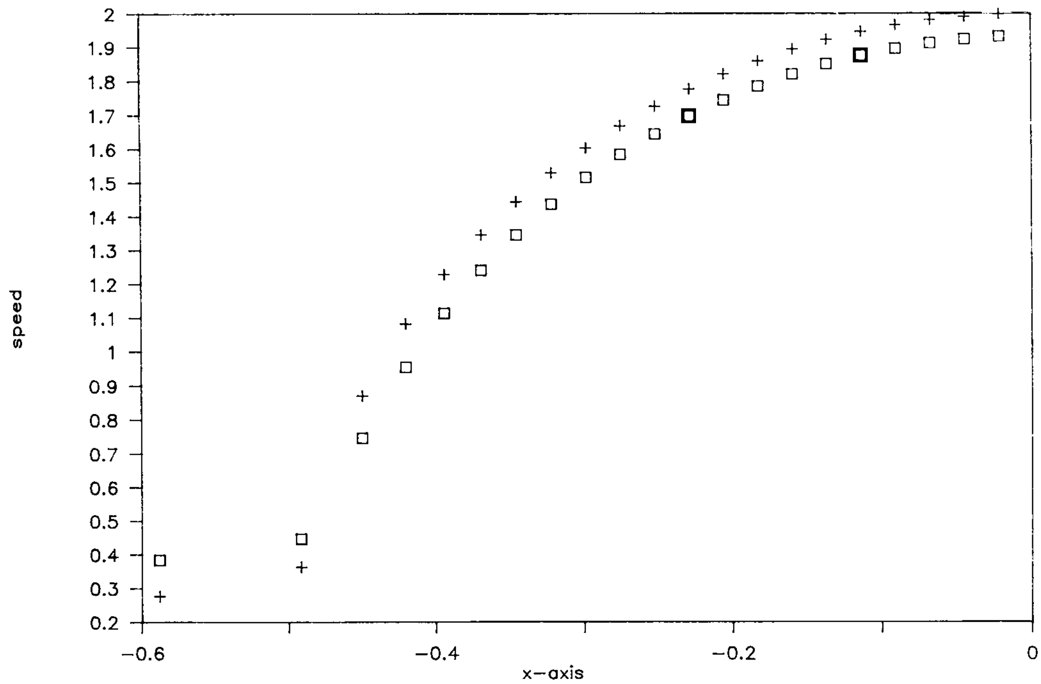

Figure 4. Speed VS. X; □ numerical solution; + exact solution

choosing a larger grid it is found best to choose a value for i_{max} and then choose a value for j_{max} which is small. By subsequently increasing j_{max} by small amounts the best possible solution is obtained. Refining the grid was attempted with a transformation of the type

$$\phi = A\exp(-B\xi^2)\tan\xi,$$
$$\psi = C\tan\eta,$$

for constants A, B and C; however this proved cumbersome and ineffective since ϕ varies with each iteration near the leading and trailing edge. This would be more effective if ϕ were known in terms of x to begin with. Another approach would be to use staggered grid spacing.

In summary, this original idea,[7] based on a philosophy of M. H. Martin[1] has produced promising results. This work comprises a portion of G. Grossman's Ph. D. dissertation.[8] It is expected that more complicated flow problems can be solved, including those with viscous, rotational and compressible effects.

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